Lecture 35

10.4 - Areas and Lengths in Polar Coordinates

Lengths

Recall that if we have a polar curve $r=f(\theta)$, we can create parametric equations:

$$x=f(\theta)\cos\theta,\quad y=f(\theta)\sin\theta$$

Thus, to compute the length of the curve from $\theta=\alpha$ to $\theta=\beta$, we have

$$L=\int_\alpha^\beta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

But, notice: $\left(r=f(\theta),\ \frac{dr}{d\theta}=f'(\theta)\right)$

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left[f'(\theta)\cos\theta-f(\theta)\sin\theta\right]^2+\left[f'(\theta)\sin\theta+f(\theta)\cos\theta\right]^2$$

$$=\left(\frac{dr}{d\theta}\right)^2\cos^2\theta\cancel{-2r\frac{dr}{d\theta}\cos\theta\sin\theta}+r^2\sin^2\theta+\left(\frac{dr}{d\theta}\right)^2\sin^2\theta\cancel{+2r\frac{dr}{d\theta}\sin\theta\cos\theta}+r^2\cos^2\theta$$

$$=\left(\frac{dr}{d\theta}\right)^2(\cos^2\theta+\sin^2\theta)+r^2(\sin^2\theta+\cos^2\theta)=\left(\frac{dr}{d\theta}\right)^2+r^2$$

So, we can simplify the formula a bit:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (r = f(\theta))$$

<u>Ex</u>: Find the length of the polar curve

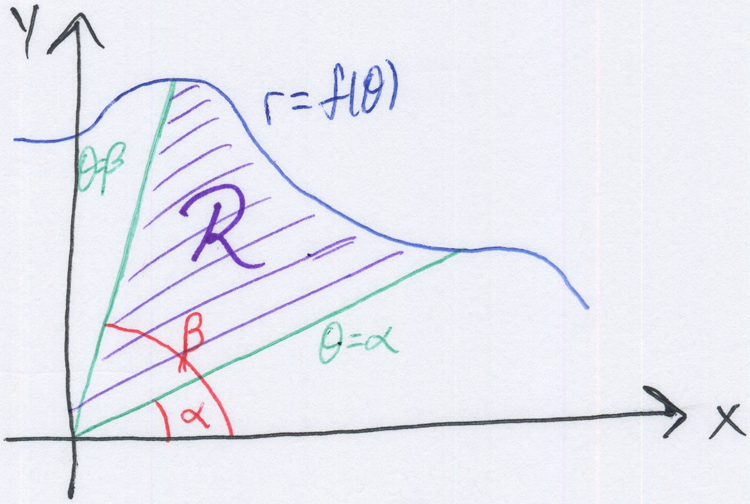
$$r = 6\sin\theta, \quad 0 \le \theta \le \pi.$$

$$\frac{dr}{d\theta} = 6\cos^2\theta$$

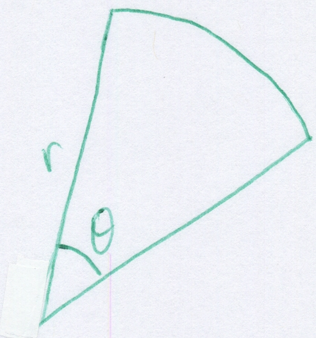
$$L = \int_0^{\pi} \sqrt{36\sin^2\theta + 36\cos^2\theta}\, d\theta = \int_0^{\pi} 6\, d\theta = \boxed{6\pi}$$

Areas

Suppose we have a polar curve $r = f(\theta)$ and we want to know the area swept out as θ goes from α to β. One such region could look like



Just as before, we want to break this region into small pieces, except this time they won't be rectangles... but rather sectors of a circle:



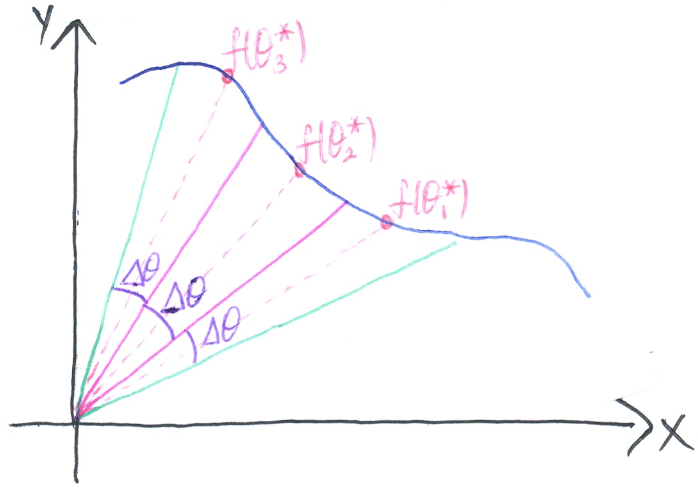
The area of this is $A = \dfrac{r^2\theta}{2}$

(This is $\frac{\theta}{2\pi}$ of a full circle, which has area πr^2.)

We divide the interval $[\alpha,\beta]$ into n subintervals of length $\Delta\theta = \frac{\beta-\alpha}{n}$, and inside each subinterval we choose a sample angle θ_i^*. Then, we can approximate the area of R as:

$$A \approx \frac{[f(\theta_1^*)]^2\Delta\theta}{2} + \cdots + \frac{[f(\theta_n^*)]^2\Delta\theta}{2} = \sum_{i=1}^{n} \frac{[f(\theta_i^*)]^2}{2}\Delta\theta$$

(n=3 in this picture)

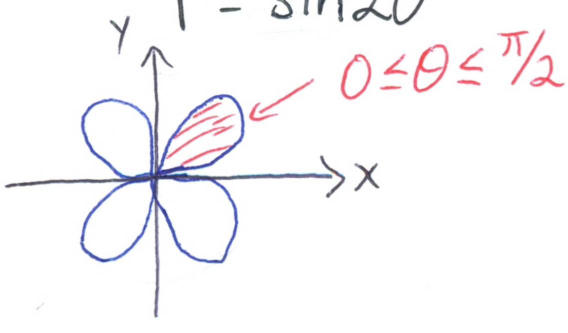


Taking $n\to\infty$ in the Riemann sum above gives:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{[f(\theta_i^*)]^2}{2}\Delta\theta_i = \int_\alpha^\beta \frac{[f(\theta)]^2}{2}d\theta = \int_\alpha^\beta \frac{r^2}{2}d\theta$$

<u>Ex</u>: Find the area inside one petal of the rose $r=\sin 2\theta$

<u>Sol</u>: <u>Step 1</u>



Step 2

$$A = \int_0^{\pi/2} \frac{\sin^2 2\theta}{2}\, d\theta = \frac{1}{4}\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/2} = \frac{1}{4}\left[\left(\frac{\pi}{2}-0\right)-(0-0)\right] = \boxed{\frac{\pi}{8}}$$

We can also find the area between to polar curves

$r=f(\theta)$ & $r=g(\theta)$

as

$$\boxed{A = \int_\alpha^\beta \frac{1}{2}\left[(f(\theta))^2 - (g(\theta))^2\right] d\theta}$$

Ex: Find the area outside the circle $r=1$ and inside the rose $r=2\sin 2\theta$.

These intersect when $1 = 2\sin 2\theta \Rightarrow \sin 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \dots \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \dots$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$ give the intersection points in the first quadrant.

We can find the area of this, and multiply by 4:

$$\text{Area} = 4\int_{\pi/12}^{5\pi/12} \frac{1}{2}\left[4\sin^2 2\theta - 1\right] d\theta$$

$$= 2\int_{\pi/12}^{5\pi/12} \left[2 - 2\cos 4\theta - 1\right] d\theta = \int_{\pi/12}^{5\pi/12} (2 - 4\cos 4\theta)\, d\theta$$

$$= (2\theta - \sin 4\theta)\Big|_{\pi/12}^{5\pi/12} = \left(\frac{5\pi}{6} + \frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{2\pi}{3} + \sqrt{3}}$$

Because polar coordinates do not provide unique reprentations of points (unless we restrict r & θ), sometimes finding the appropriate θ-values is a little tricky.

Ex: Find the θ-values at which

$$r = 2\cos 2\theta \quad \text{and} \quad r = 1$$

intersect.

① $2\cos 2\theta = 1$

$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$

$\theta = \frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi$

② $2\cos 2\theta = -1$

$\cos 2\theta = \frac{-1}{2}$

$2\theta = \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi$

$\theta = \frac{\pi}{3} + n\pi, \frac{2\pi}{3} + n\pi$

Intersects at: $\theta = \frac{\pi}{6} + n\pi, \frac{\pi}{3} + n\pi, \frac{2\pi}{3} + n\pi, \frac{5\pi}{6} + n\pi$